# Pursuing the Ubiquitous Pendulum

*Art Stinner* and *Don Metz*

**All of physics comes from Galileo's inclined plane.** *(Anon)*

**It is not too much to assert that without the pendulum there would have been no Principia.** *(Richard Westfall)*

The pendulum and the inclined plane played a central role in the development of kinematics and dynamics during the 17th century. Galileo's kinematics rests on experiments and thought experiments connected with the inclined plane and the pendulum. Later, Newton used pendula and Galilean kinematics to test and demonstrate his laws of motion and to confirm the equivalence of inertial and gravitational masses.[1,2]

The pendulum also played an important role in the next two centuries. George Atwood used the pendulum in his famous machine named after him to test Newton's second law of motion. Count Rumford, mentioned in textbooks as the debunker of the caloric theory, in 1781 adapted the pendulum in his ballistic device to measure the muzzle velocity of bullets, used until the recent effective application of high-speed photography. Jean Foucault designed a very long and heavy pendulum to demonstrate for the first time directly that the Earth revolves around its axis. Indeed, the study of the harmonic oscillator in all its manifestations in dynamics and electricity, in our everyday experiences and our models on the atomic level, can be traced back to the properties of the pendulum. Recently the pendulum again became a high-profile object in the demonstration of chaos theory.

Here we offer three examples, inspired by Galileo's work, for connecting the physics of the pendulum with free fall and the inclined plane. The first example discusses the time it takes for a ball to slide down a frictionless inclined plane along a chord, drawn from the lowest point of a vertical circle to any point on the circle (see Fig. 1). The second example discusses a timing method that Galileo tested in measuring free fall directly (see Fig. 2), and the third example outlines an approach we have developed for accurately calculating the period of a pendulum for any angle, using only the kinematics of accelerated motion along an inclined plane, first studied by Galileo (see Figs. 3 and 4).

## The Pendulum and Free Fall

Textbooks often tell students that Galileo "diluted gravity" using an inclined plane and extrapolated to the motion of free fall. However, textbooks generally don't mention that Galileo also studied free fall directly, using an ingenious method of timing the fall with a pendulum. Before repeating his experiment to test free fall directly, we will describe his arguments to show that the time of descent of an object (on a frictionless surface) along an incline represented by any chord is the same, and is equal to the time it would take for an object to fall through a distance of $2L$ (see Fig. 1).

Galileo proved this theorem, using a geometric argument with Euclidean ratios $T_1^2 / T_2^2 = D_1 / D_2$ Using modern algebraic methods is much easier: Referring to Fig. 1, we first note that two chords that connect the endpoints of the diameter of a circle are always perpendicular (i.e. AD $\perp$ DE, AC $\perp$ CE, and so

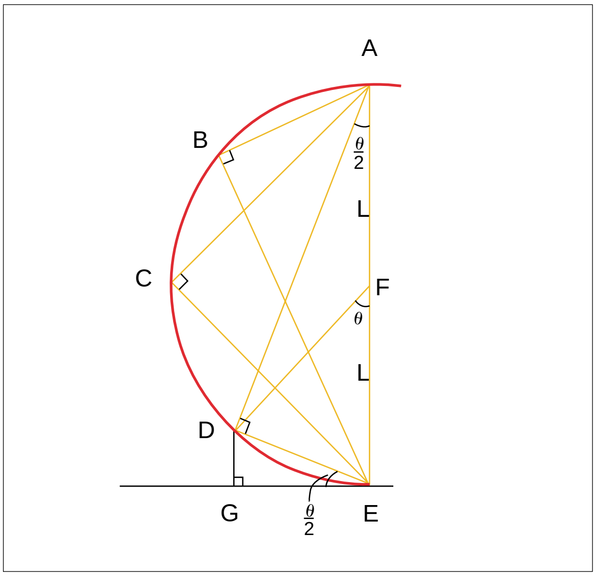


**Fig. 1. Galileo's geometric argument for equal time descent.**

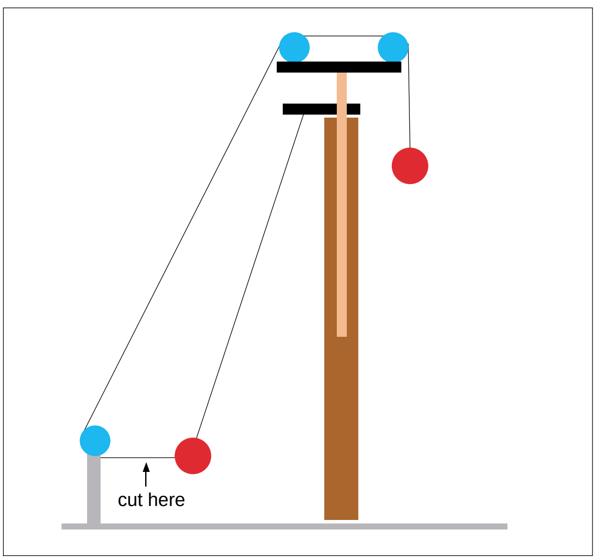


**Fig. 2. The free-fall experiment using a pendulum.**

on). We also note that by comparing angles of triangles that angle DAF is $\theta/2$ and in turn this is the angle the incline makes with the horizontal (DEG). Therefore, for a sphere sliding down the incline DE, the acceleration is $g \sin \theta/2$ and from triangle ADE, $\sin \theta/2 = d/2L$, where $d$ is the length of the incline and $L$ is the radius of the circle. It follows from $d = \frac{1}{2}at^2$ that the time for the sphere to roll down the plane is $2(L/g)^{1/2}$. Now, if we consider a freely falling sphere released from point A, the distance it will fall can also be found from $d = \frac{1}{2}at^2$ and the time for free fall equals $2(L/g)^{1/2}$. Clearly, the argument holds for all angles of $\theta$ between 0° and 180°.

Our second example is based on an experiment Galileo actually performed as a young man in about 1600.[3] A pendulum was held out from a vertical board and released simultaneously with the dropping of a weight. He adjusted the pendulum length until the "thud" of the pendulum hitting the wall coincided with the "thud" of the weight hitting the floor. Since the period of the pendulum was known, the time it took for the pendulum to hit the wall would be one-fourth of the period of the pendulum. We have built a simple device (see Fig. 2) to perform an experiment in class to show that Galileo also tested free fall other than using an inclined plane. Two identical spheres, one attached as a pendulum and the other hanging freely, are connected with a fine line by a pulley system as shown. We can adjust the height of the suspended ball on the right-hand side of the apparatus such that it strikes the ground at the same time as the pendulum contacts the support when the line is cut. Then, the time for free fall exactly equals one-fourth of the period of the pendulum. It is now easy to find the value of $g$ from $d = \frac{1}{2}gt^2$, where $d$ is the measured

height of the freely falling sphere above the ground. (Students should be aware of the fact that the accuracy of the period of a pendulum, as given by Huygens' formula, is acceptable only for angles less than about 10°).

We should remember that Galileo stated the period of a pendulum only as a proportionality statement. He said in his book, published in 1638, just before his death,

> *"As the times of vibrations of bodies suspended by threads of different lengths, they bear to each other the same proportion as the square roots of the lengths of the thread."* [4]

Therefore, Galileo was not able to use the pendulum directly to determine the value of $g$, the acceleration due to gravity. Huygens, on the other hand, about 30 years later, was able to find the formula we use today [$T = 2\pi\, (L/g)^{1/2}$] and solve for $g$ in terms of the period $T$ and the length $L$ of the pendulum. He obtained very accurate (three significant figures) answers.[1] It should also be noted, however, that Galileo was not interested in finding the value of $g$, but rather "the ratio of the distance fallen from rest to the length of a pendulum whose period divided by four is the time required for that fall." Students could try to show that, at the risk of committing a slight anachronism, using Huygens' formula $T = 2\pi (L/g)^{1/2}$, this ratio is $\pi^2/8$, and that the ratio is constant for all heights of fall, at all places on Earth and even on the Moon.

## A General Solution for Calculating the Period of a Pendulum

Galileo's biographer, Vincenzo Viviani, recalls that Galileo became interested in the motion of a pendulum after he observed a suspended chandelier swing back and forth in a cathedral. The astute Galileo recognized the practical implications for motion that was isochronic. At this time, the mechanical clock, using a heavy weight and gravity to power the mechanism, worked in an irregular and unpredictable manner. A pendulum with its even and natural motion, connected to the escape mechanism of a clock, could be used to beat out equal time intervals, substantially improving the timing devices of the day.

Surprisingly, Galileo mistakenly believed that the period of a pendulum was independent of the amplitude of the swing. He states that:

> *"If two people start to count the vibrations, the one the large, the other the small, they will discover that after counting tens and even hundreds, they will not differ by a single vibration, not even by a fraction of one."* [5]

In 1639, the mathematician Mersenne, who also studied the theory of the pendulum, argued against Galileo and claimed that the period of the pendulum was not isochronous:

> *"And if he (Galileo) had merely counted the thirty or forty oscillations of the one pulled aside twenty degrees or less and the other eighty or ninety degrees, he would have known that the one having the shorter swings makes one oscillation more in thirty or forty oscillations."* [6]

How could Galileo have tested his claim that the period of a pendulum is independent of the amplitude? Clearly, direct comparison of the swings of two pendula, with different starting amplitudes, should show that they will be out of phase very quickly, as Mersenne claimed. So, why was he still convinced of his claim? One answer is that he believed in the constancy of the period of the swing so strongly that he dismissed the observation on physical grounds (friction?), even though the two pendula were out of phase after a number of swings.

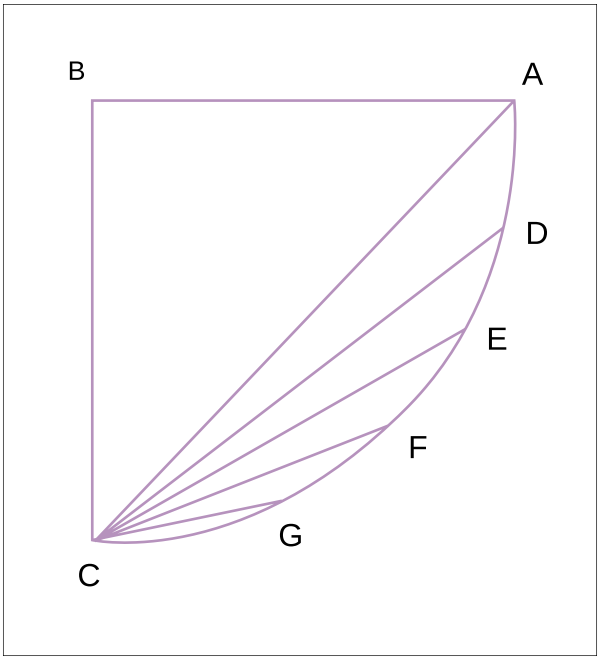


**Fig. 3. Galileo's sketch taken from *Two New Sciences*.[8]**

Galileo also experimented with balls rolling along a polished concave hoop. Describing the motion along the hoop he stated that:

> *"...wherever you place the ball , whether near to or far from the ultimate B, and let it go, it will arrive at point B in equal times ..."* [3]

Together with his law of chords, his law of amplitude amounted to believing that the shortest time of descent of a frictionless object in a vertical plane (brachistochrone) is the arc of a circle. Actually, Galileo did investigate the motion of a ball along an inclined plane as compared to the motion of a ball along the corresponding arc of the circle (see Fig. 3). His conclusion, after a lengthy geometric argument, was that the motion along the longer arc is such that it takes less time to reach the nadir than for the motion

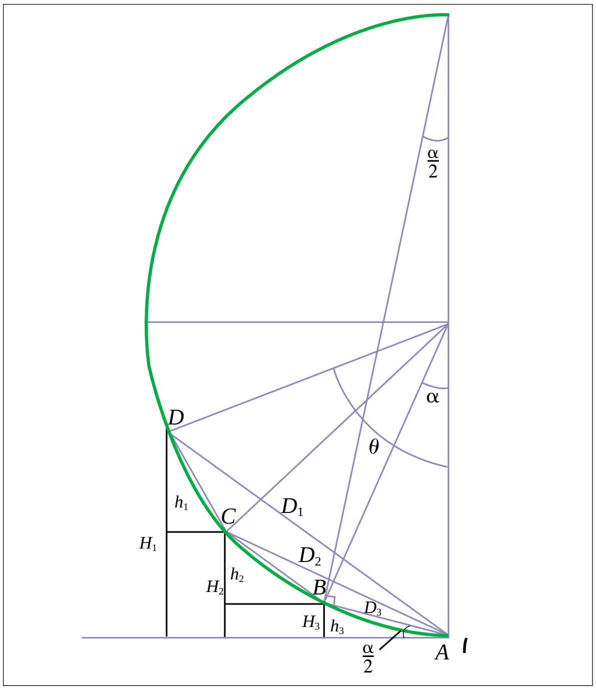


**Fig. 4. Sketch for *n*-inclined planes calculations.**

along the shorter inclined plane. It is interesting to note that already 100 years earlier, Leonardo da Vinci came to the same conclusion.[7] Later, Huygens showed that it is the cycloid and not the arc of a circle that is the least time of descent of an object in a vertical plane and applied this finding to improving the pendulum clock.

We will now investigate the time of descent of a ball along the arc of a circle by placing a number of inclined planes along the arc, as shown in Fig. 4. The more inclined planes we use, the more closely we will mirror the motion of a ball along the arc. We will find that the time of descent does depend on the angle of amplitude and two results will surprise us. One is that it takes only about 10 inclined planes to get accurate results. The other unexpected result is that the period can vary to a maximum of about 17% from that predicted by Huygens' formula.

We are now in the position to apply Galileo's findings and the kinematics of uniform acceleration to determine the period of a pendulum to any desired accuracy, for any angle. We imagine a sphere rolling down a number of inclined planes. The motion starts from rest, then accelerates uniformly to velocity $v_1$. This velocity now becomes the initial velocity at the start of the second inclined plane, again accelerating uniformly (with a lower acceleration) to a velocity of $v_2$, etc., until the sphere reaches the lowest point. To find the period of our pendulum, we will add all of the individual times for each incline for one-quarter of a swing and multiply by four. The initial velocity of the sphere on each plane is the final velocity of the sphere on the preceding plane; therefore; we can apply

$$v_2{}^2 = v_1{}^2 + 2ad \tag{1}$$

to calculate the velocities. The acceleration for each plane is given by $a = g \sin \theta$ such that $\sin \theta = h/d$, where $h$ is the height and $d$ is the length of the plane. We also know that for a uniformly accelerated object,

$$d = v_1 t + \frac{1}{2} a t^2. \tag{2}$$

Solving this equation for $t$ using the quadratic formula gives us the time it takes to descend the plane:

$$t = \frac{-v_1 \pm \sqrt{(v_1{}^2 + 2ad)}}{a}. \tag{3}$$

A spreadsheet is well suited for making repeated calculations and has the additional advantage of letting us quickly see the effects of any changes that we might make to the situation. In our case we wish to calculate the period of a pendulum for any angle using any number of inclined planes to approximate the arc of the pendulum. The greater the number of inclined planes the more accurate the result. Since we choose the angle of swing, we need to find the height ($h_k$) each plane rises and the length of the planes ($d$) to complete the calculations.

For convenience we select an arc of radius 1, let θ be the angle of the swing, and let $n$ be the number of inclined planes used to calculate the period (see Fig. 4). Thus, the angle the last inclined plane subtends is $\alpha = \theta/n$. For example, Fig. 4 illustrates an angle of swing of 75° with $n = 3$ inclined planes, such that the last inclined plane subtends an angle of 25°. In this case, the angle each chord segment subtends is 25°, 50°, and 75°, respectively. We note that by comparing angles of triangles that angle AFB will be $\alpha/2$; therefore, the length of the plane ($D_3$) from right triangle

**Table I. Calculating the period of a pendulum for large angles.**

| A1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| B | | 60.00 | 0.1047 | | | |
| C | N | Angle (θ) | *d* | | | |
| D | 10 | 0.10 | 2 sin ($a$/2) | | | |
| E | | | | | | |
| F | $n$ | $D_k$ | $H_k$ | $h_k$ | $v_k$ | $t_k$ |
| G | | $D_k = 2\sin[(n–k+1)(\alpha/2)]$ | $H_k = D_k \sin[(n–k+1)(\alpha/2)]$ | $h_k = H_k – H_{k+1}$ | $v_k = [–(v_{k–1})^2 + 2gh_{k–1}]^{1/2}$ | $t_k = \frac{-vk-1 \pm \sqrt{v_k-1)^2+gh_k}}{g(h_k/d)}$ |
| | | | | | | |
| | | | | | | |
| H | | | | | 0 | |
| i | 1 | 1.00 | 0.50 | 0.09 | 1.31 | 0.1596 |
| I | 2 | 0.91 | 0.41 | 0.08 | 1.82 | 0.0668 |
| J | 3 | 0.81 | 0.33 | 0.07 | 2.18 | 0.0523 |
| K | 4 | 0.72 | 0.26 | 0.07 | 2.46 | 0.0451 |
| L | 5 | 0.62 | 0.19 | 0.06 | 2.68 | 0.0407 |
| M | 6 | 0.52 | 0.13 | 0.05 | 2.85 | 0.0379 |
| N | 7 | 0.42 | 0.09 | 0.04 | 2.97 | 0.0360 |
| O | 8 | 0.31 | 0.05 | 0.03 | 3.06 | 0.0347 |
| P | 9 | 0.21 | 0.02 | 0.02 | 3.11 | 0.0339 |
| Q | 10 | 0.10 | 0.01 | 0.01 | 3.13 | 0.0335 |
| R | | | | | | |
| S | | | | | $1/4T_N$ | 0.5405 |
| T | | | | | $T_n$ | 2.162 |

**Table II. Comparing the periods of a pendulum for the three approaches discussed.**

$T_{Modern} = 2\pi(L/g)^{1/2}\,[1 + 1/4 \sin^2(\theta/2) + 9/64 \sin^4(\theta/2) + 225/2304 \sin^6(\theta/2)$

$T_{Galileo}$ = Based on spreadsheet calculations for $n = 10$

$T_{Huygens} = T = 2\pi\,(L/g)^{1/2}$

| θ | $T_{Hugyens}$(s) | $T_{Integral}$(s) | $T_{Galileo}$(s) |
|---|---|---|---|
| 5° | 2.01 | 2.01 | 2.02 |
| 10° | 2.01 | 2.01 | 2.02 |
| 30° | 2.01 | 2.04 | 2.05 |
| 60° | 2.01 | 2.15 | 2.16 |
| 90° | 2.01 | 2.35 | 2.37 |

ABF is 2 sin ($\alpha$/2). In order to find $h_k$ we need to find the cord length ($D_k$) and the height of this chord above the nadir ($H_k$). Using triangle ADE, we can show that $D_k = 2\sin[(n–k–1)(\alpha/2)]$ and therefore $H_k = D_k \sin[(n–k+1)(\alpha/2)]$, where $k$ = 1,2,3 ... and the angle of each inclined plan is $[(n–k+1)(\alpha/2)]$. The height of a plane can be calculated by $h_k = H_k – H_{k+1}$. As a result, Eq. (1) becomes $v_k = [–(v_{k-1})^2 + 2gh_{k-1}]^{1/2}$ and the time for the ball to move down the $k$th incline can be found by solving Eq. (2) using the quadratic formula. Thus,

$$t_k = \frac{-v_k - 1 \pm \sqrt{(v_k - 1)^2 + gh_k}}{g(h_k/d)}$$

and the corresponding period would be $T = 4(t_1 + t_2 + t_3 + \ldots\ t_n)$.

The layout of the spreadsheet is shown in Table I. Row G contains the formulae that are used to calculate the values in the corresponding columns. We can easily select the angle of the swing by entering a value in cell B3. In this example we show the results for an angle of 60° and $n$ = 10 inclined planes. The problem of finding the period of a pendulum for any angle (up to 90°) can also be solved analytically using elliptical integrals.[2] In Table II we compare the results of our model for finding the period of a pendulum with those based on Huygens' formula and the modern analytical solution using elliptical integrals for a length of 1.0 m.

## Concluding Remarks

Our first example is an exercise that challenges physics students. When encountering this theorem for the first time, it often seems counterintuitive, much like the discovery that the descent of a ball along a bowl is quicker than along the corresponding inclined plane. Moreover, proving the theorem involves the combining of geometry, trigonometry, algebra, and physics. The second example always delights students and teachers alike for its ingenuity and novelty (textbooks simply do not mention it).

The third example illustrates the number-crunching power of the PC in a way that students can easily understand. Our method of approximating the motion of a pendulum with the continuous rolling sphere on $n$ inclined planes with incrementally decreasing accelerations yields accurate results even when compared to our most comprehensive formula, which in itself is an approximation, based on elliptical integrals. Indeed, our derivation is quite easy to understand, even for the student of introductory physics.

***A final note***: Originally, we were going to have a fourth example, namely the Galilean theorem, showing that the speed of a pendulum at the lowest point is directly proportional to the arc length. This was an important theorem for Newton when demonstrating the third law of motion, using colliding pendula, what really is a demonstration of *the conservation of linear momentum* principle. However, this has been well done by Ehrlichson in a recent issue of *TPT*.[2]

### References


1. A. Stinner, "Linking *The Book of Nature* and *The Book of Science*," *Sci. & Educ.* **10**, 323–344 (2001).
2. H. Ehrlichson, "A proposition well known to geometers," *Phys. Teach.* **39**, 152–153 (March 2001).
3. M. Matthews, *Time for Science Education: How Teaching the History and Philosophy of Pendulum Motion Can Contribute to Science Literacy* (Kluwer Academic Publishers, 2000), p. 107.
4. See Ref. 3, p. 96.
5. See Ref. 3, p. 113.
6. See Ref. 3, p. 115.
7. See Ref. 3, p. 87.
8. Galileo Galilei, *Two New Sciences,* translated by Henry Crew, Dover (Dover, 1954/1637)..

For more information, refer to the International Pendulum Project (IPP): www.arts.unsw.edu.au/pendulum.



**Arthur Stinner** *has a doctorate in science education from the University of Toronto and a M.S. degree in physics from York University. He has taught high school and college physics for more than 20 years and has been a professor of science education at the University of Manitoba since 1989. He has published widely, and his research interests include the designing of large-context problems and using the history of science in the teaching of physics.* *University of Manitoba, Winnipeg, MB R3T 2N2, Canada; stinner@cc.umanitoba.ca*

**Donald Metz** *has a B.S. in physics, an M.Ed. in mathematics and natural sciences with an emphasis in computer applications, and a Ph.D. in science education. He taught high school physics for more than 20 years and remains actively involved in the professional development of physics teachers. Currently, he is an assistant professor in the science education program at the University of Winnipeg.* *University of Winnipeg, Winnipeg, MB R3B 2E9; d.metz@uwinnipeg.ca*