

Calculating the age of the Earth and the Sun

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Abstract

Calculations at high school level can be used to show how Newton estimated the age of the Earth from the cooling of a hot body and later Helmholtz and Kelvin applied thermodynamic arguments to finding the ages of both the Sun and the Earth. Finding the Sun to be younger than the Earth was a puzzle until the discovery of radioactivity and the Sun's true source of power.

This paper will review the main attempts made to calculate the age of the Earth and the Sun, beginning with Newton's thought experiment and ending with Hans Bethe's thermonuclear model of the Sun's energy. These ideas will be presented in two parts. In Part I, special attention is paid to the protracted debate about the age of the Earth in the second half of the nineteenth century that involved Kelvin and Helmholtz. The first part of the paper will terminate with a brief mention of the radioactive/nuclear theories being developed just prior to the death of Kelvin in 1907. Part II will discuss the rise of radioactive dating and the emergence of the thermonuclear model. The results of the calculations are given in the main text but details can be found in the appendices, called Blocks. Most of the problems discussed can be used in senior high school physics in North America or at the A-level in Britain. The content of the paper should also provide a good background for physics instructors to enrich their presentations with interesting problems beyond the textbook.

Newton's thought experiment

Newton presented a thought experiment in the *Principia* to show that a large body like the Earth made of molten iron would take about 50 000 years to cool. He first estimated the time it would take to cool for a 'globe of iron of an inch in diameter, exposed red hot to open air'. He then argued

that, since the heat retained is in proportion to the volume and the heat radiated in proportion to the exposed area, the time for cooling would be proportional to the diameter (Dalrymple 1991, p 28). See Block A1.

We can check how good a guess Newton made by using Stefan's radiation law of 1878, and not, as may be expected, Newton's law of cooling. Textbooks generally state that Newton discovered experimentally that the rate of cooling of an object is proportional to the difference between the temperature of the object and that of its surroundings. This proportionality statement leads to an exponential expression that most physics textbooks discuss, and students use it to solve problems. It turns out that Newton used the proportionality statement only to calibrate a linseed thermometer that could measure temperatures higher than 200 °C (French 1993). Moreover, it is now known that this relationship does not hold for high temperatures (Silverman 2000).

Using Stefan's law of radiation to calculate the cooling time

The law states that H , the rate at which an object radiates heat per unit area is proportional to the fourth power of the absolute temperature:

$$H \propto T^4 \quad \text{or} \quad H = e\sigma T^4.$$

σ is a universal constant and found experimentally to be $5.7 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$, and e is a constant between 0 and 1, depending on the metal considered.

The cooling time from an initial temperature T_i to a final temperature T_f is given approximately by

$$t = (1/T_f^3 - 1/T_i^3)mc/3\sigma A$$

where T_i is the temperature at time t , T_f the ambient temperature (both expressed in kelvins), m is the mass, c the specific heat capacity and A the area of the object (see Block A2).

Applying this formula to the small iron sphere, we find that the cooling time is about 47 minutes. For the time of cooling of a globe of the size of the Earth that is made entirely of iron, we obtain about 45 000 years (see Block A2).

This is in very good agreement with Newton's conclusion. Of course, the cooling time for such a large object would be very much longer, because it would take considerable time for the heat to be conducted to the surface as the body cools. To solve the problem of temperature distribution for such a case would require the application of a Fourier analysis, later accomplished by Kelvin.

Count de Buffon tests Newton's thought experiment

Buffon was one of the most productive scientists of the eighteenth century. He was interested in determining the age of the Earth, and being a wealthy man he asked his foundry to make him ten iron spheres, in increments of $\frac{1}{2}$ inch up to 5 inches. He concluded that if the Earth had been made of molten iron, it would require 42 964 years to cool below incandescence and 96 670 years to cool to the present temperature.

Today, we smile at such efforts. However, one must remember that in the mid-eighteenth century alchemy was still in vogue and there was not even an elementary theory of heat established. Buffon believed and demonstrated that nature was rational and could be understood through physical processes. He was also the first to apply experimental techniques to the problem of the age of the Earth. A century passed, however, until Helmholtz and Kelvin, equipped with more sophisticated physics and experimental procedures, tackled the problem again.

Helmholtz and the age of the Sun

Hermann von Helmholtz was the most famous German natural philosopher and cosmologist of his generation. He was one of the original contributors to the General Principle of the Conservation of Energy. Already in 1854 he argued that the Sun's energy must be supplied by gravitational contraction because no known chemical reaction could produce sufficient energy. His calculations showed that the Sun could supply the energy we measure without us being aware of the contraction. According to his model the Sun could be about 20 million years old.

The Sun as a furnace burning coal

Helmholtz first calculated that if the Sun's energy were due to a chemical source, the life expectancy would be about 5000 years. This is easy to show and can be discussed in a science class at the upper middle school level (see Block B1).

The Sun as a gravitationally collapsing body

Helmholtz immediately rejected this model and argued that gravitational collapse of the original cloud of material to the present size of the Sun was the source of the Sun's size and its continued production of energy. He assumed that material fell into a proto-mass from infinity and the Sun grew by accretion and heated up to the present temperature.

In his famous 'Popular Lectures' of 1857 he discussed this model and gave the value for the gravitational potential energy of the Sun as shown in Block B2. He found that the approximate gravitational potential energy then is 2.3×10^{41} J. Since he knew the energy output of the Sun, Helmholtz was able to estimate the age of the Sun and found it to be about 21 million years (see Block B3).

The temperature of the Sun

Helmholtz estimated the temperature of the Sun by assuming that the specific heat capacity of the Sun was equal to that of water and that the mechanical equivalent of heat (just recently published by Joule) was about 4.2 joules per calorie, as shown in Block B4. The temperature Helmholtz found was 28 611 000 °C. It is interesting to note that the estimated temperature of the centre of the Sun is about 10^8 K.

The pressure at the centre of the Sun

Helmholtz also calculated the pressure at the centre of the Sun and obtained a value of about $1.3 \times 10^{14} \text{ N m}^{-2}$. To show this, only a very simple calculation is needed—see Block B5. The modern value for the pressure at the centre of the Sun is about 10^{15} N m^{-2} , an order of magnitude different. Of course, Helmholtz did not realize that at that pressure and temperature thermonuclear fusion would be initiated.

The rate of shrinkage of the Sun

Finally, Helmholtz estimated the shrinkage of the Sun necessary to produce sufficient energy to account for the estimated output of 3.6×10^{26} joules per second. He came to the conclusion that about 80 m of shrinkage per year is necessary. The results of the calculations made by Helmholtz can be studied in Block B6.

Kelvin and the age of the Sun

Kelvin first established the age of the Sun, before calculating the age of the Earth. Like Helmholtz he quickly dismissed the chemical energy theory, because it allowed for less than 10 000 years. He then investigated the physics of his meteoric hypothesis. In this hypothesis he assumed that the energy of the Sun is replenished by a constant rate of meteoric bombardment.

He calculated the kinetic energy of impact of a mass of 1 pound of matter falling into the Sun at the escape velocity of the Sun. Using SI units, a quick calculation shows that a 1 kg mass hitting the Sun at the escape velocity of 624 km s^{-1} would have a kinetic energy of $\frac{1}{2} \times 1 \times (6.24 \times 10^5)^2 \text{ J}$, or $1.94 \times 10^{11} \text{ J}$.

A simple calculation then showed that about 1/5000 of the Sun's mass over a period of 6000 years would suffice to account for the energy given off by the Sun. It must be remembered that this would represent about 70 times the mass of the Earth! We can quickly check this.

The Sun's energy output was found to be about $3.6 \times 10^{26} \text{ J s}^{-1}$. Therefore in 6000 years time the Sun will have liberated $6000 \times 3.15 \times 10^7 \times 3.6 \times 10^{26}$ or $6.8 \times 10^{37} \text{ J}$. Dividing this by the kinetic energy of a 1 kg mass falling in we get $6.8 \times 10^{37} / 1.94 \times 10^{11}$ or $3.5 \times 10^{26} \text{ kg}$. This is $3.5 \times 10^{26} / 2 \times 10^{30}$, or 1/5000 of the mass of the Sun. However, by about 1861, Kelvin rejected this theory also, because

1. There was no spectroscopic evidence found that anything faster than about 1/20 of the escape velocity of the Sun is found in the vicinity of the Sun.
2. The effect of the additional mass of the Sun on the period of rotation of the Earth would have been detected.

Kelvin calculated this effect and found that it would be about 1/8 of a year in a 2000 year period. His calculations are quite sophisticated but we can check his value with a relatively simple approach, using Kepler's third law and Newtonian mechanics. Kelvin then argued that such a discrepancy would have been found and therefore he finally accepted Helmholtz's theory of gravitational contraction as the only viable one (see Block C1).

Kelvin's commitment to the gravitational model of Helmholtz emerged only gradually, starting with his early recognition of the soundness of Helmholtz's ideas after reading his lecture of 1857 to the time he developed his model for the cooling of the Earth in 1862. At the beginning he thought that his meteoric theory and Helmholtz's gravitational collapse theory would complement each other, but by 1862 he grudgingly deferred to Helmholtz. Kelvin did calculate the age of the Sun, using both a linear and an exponentially decreasing density. He found that for the first case the age of the Sun was 20 million years and for the second about 60 million years. So he was able to push the age of the Sun to three times the value based on Helmholtz's simple model. Kelvin's models predicted that a gravitational shrinkage rate was very close to that calculated by Helmholtz. It is interesting to note that he estimated that about 80% of the mass of the Sun was contained inside a sphere of half the radius.

Kelvin calculates the age of the Earth

Having established that the Sun is at least 20 million years old, and probably as old as 60 million years, Kelvin set out to determine the age of the Earth. His seminal paper 'On the Secular Cooling of the Earth' was published in 1862 and produced an instant sensation in both scientific and public domains. Kelvin made the following assumptions, as recorded in his paper:

1. Most of the Earth's heat was originally generated by gravitational energy.

2. The Earth cooled from a temperature of about 3700 °C to the present temperature of about 0 °C very quickly, probably in about 40 000–50 000 years.
3. The average temperature of the Earth's surface has not changed significantly since then.
4. The interior of the Earth is solid and therefore only conduction is significant.
5. In all parts of the Earth a gradually increasing temperature has been found in going deeper. This finding implies a continual loss of heat by conduction.
6. Since the upper crust does not become hotter from year to year, there must be a secular loss of heat from the whole Earth.

The physical constants that he needed were:

1. The temperature gradient of the surface of the Earth.
2. The specific heat capacity of the Earth's crust.
3. The thermal conduction coefficient for the crust.

Kelvin estimated an average for the temperature gradient to be (in modern units) 1/28 °C per metre of depth and decided that the thermal conduction coefficient of the Earth's crust was about 0.8 W m⁻¹ K⁻¹ (see table 1). Most of these data were based on the specific heat capacities of three Edinburgh rocks and the temperature measurements were made in mines in Scotland. Kelvin admitted that one was ignorant about the effects of high temperatures altering the conductivities and specific heat capacities of rocks, as well as their latent heat of fusion. He stated that:

We must, therefore, allow very wide limits in such an estimate as I have attempted to make, but I say that we may with much probability say that the consolidation cannot have taken place less than 20 000 000 years ago, ... nor more than 400 000 000 years...

(Kelvin 1890, p 300)

Kelvin assumed that the temperature after solidification was 3700 °C at the centre and about 0 °C at the surface. These temperatures, he argued, remained constant over millions of years and were located on the two sides of an arbitrary infinite plane in an infinite solid. Such a distribution provides the initial conditions for the discontinuity between the two planes. In his own words, he then:

applied one of Fourier's elementary solutions to the problem of finding at any time the rate of variation of temperature from point to point, and the actual temperature at any point, in a solid extending to infinity in all directions, on the supposition that at an initial epoch the temperature has had two different constant values on the two sides of a certain infinite plane

(Kelvin 1890, p 301)

(See Block C for details about Kelvin's calculations.)

Based on the specifications given Kelvin derived two equations:

$$v(x, t) = \frac{2V}{\pi^{1/2}} \int_0^{x/2\sqrt{kt}} dz e^{-z^2} \quad (1)$$

and

$$\frac{dv}{dx} = \frac{V}{(\pi kt)^{1/2}} e^{-x^2/4kt}. \quad (2)$$

The complete solution to equation (1) is

$$v(x, t) = V \operatorname{erf}[x/2(kt)^{1/2}] \quad (3)$$

where $\operatorname{erf}(x) = (2/\pi^{1/2})(x - x^3/3 + x^5/10 - x^7/42 \dots)$. Here 'erf' is the error function and in calculations is best obtained from tables (see Block C).

Equation (1) allowed Kelvin to determine the temperature as a function of depth and time and equation (2) gave him the value of the temperature gradient as a function of depth and time (see Blocks C3 and C4).

Using equation (2), Kelvin showed that if

- (a) the time is 1000 million years or less, then the temperature gradient is essentially zero below about 600 miles.
- (b) the time is about 100 million years, we have the present gradient profile of 1 °C per 28 m, remaining constant to a depth of about 30 km.

Therefore, he concluded that the Earth is probably about 100 million years old. He showed that this gradient value should stay pretty well constant to a depth of about 30 km. At 160 km the temperature is constant at about 3700 °C (see figure 3).

In his paper he drew the following conclusions (converted to SI units):

Table 1. Physical measurements.

	Kelvin	Today
Temperature gradient of the Earth's crust	36 °C km ⁻¹	30 °C km ⁻¹
Heat flow through the Earth's crust	0.8 W m ⁻²	1 W m ⁻²
Thermal conduction coefficient of the Earth's crust	0.8 W m ⁻¹ K ⁻¹	1 W m ⁻¹ K ⁻¹

1. The limits of the Earth's age are between 20 million and 400 million years.
2. The temperature gradient will remain constant at about 1/28 m per °C for about 30 km.
3. During the first 1000 million years the variation of temperature does not become 'sensible' at depth exceeding 910 km.
4. The temperature gradient diminishes in inverse proportion to the square root of the time (see equation (2)).

Finally, it is interesting to compare the physical parameters used by Kelvin to those that geologists accept today (table 1). Kelvin would be pleased and would probably say that no major changes are necessary in his model.

The decline of the reign of limited time

The physicists' models of the age of the Earth and the age of the Sun, as epitomized by Kelvin's and Helmholtz's calculations, reigned supreme until almost the end of the century. These models set strict limits on the age of the Earth. Geologists and biologists were generally not sufficiently well trained in mathematics and had inadequate understanding of the new science of thermodynamics to challenge Kelvin and the physicists in setting these strict limits. The physicists, on the other hand, were not acquainted with the methods and measurements made by the geologists and generally had a low opinion of their attempts to emulate the empirical methods of physics.

The first serious challenge came in 1890 from John Perry, an accomplished mathematician and engineer, a former student of Kelvin. He argued that for the last 30 years Kelvin's calculations had been repeated by schoolboys as an abstract mathematical exercise. Having concentrated on the mathematical details only, the assumptions and preconceptions, 'especially those simplifying assumptions to facilitate calculations', had been lost (Burchfield 1975, p 135).

He went on to show that if we assumed that the Earth's conductivity were not homogeneous, but greater near the centre by a factor of 10, the cooling time would be increased by a factor of 56. In addition, he argued that if some degree of fluidity exists in the Earth, then thermal conductivity must be supplemented by convection.

Kelvin took Perry's argument seriously and responded, finally conceding that it may be possible, after all, to extend the age of the Earth to as much as 1000 million years and not violate the laws of physics. But he was quick to add that the Sun's heat placed a great limitation on the age of the Earth. In fact, Kelvin reminded Perry that the recent (1877) calculations made by the noted mathematical physicist P G Tait limited the age of the Sun to only 10 million years.

Perry suggested that perhaps there may be other sources of energy than gravitational contraction. Indeed, there were many mechanisms and theories presented in the 1890s that were supposed to account for the vast energy of the Sun, other than gravitational. Unfortunately, upon close scrutiny, they all violated the laws of thermodynamics.

Perry managed to take away the feeling of mathematical certainty from the physicists' arguments and expose the underlying weakness of their assumptions and the inadequacy of the physical data they used. Meanwhile, the mass of evidence accumulated by the geologists that was based on improved empirical methods began to convince them that the physicists must be wrong about the age of the Earth. Geologists became bolder and more confident about their methodology and slowly freed themselves from the domineering influence of the physicists.

New energy sources in the 'storehouse of creation'

In July 1903, W E Wilson, an independent gentleman-astronomer with a longstanding interest in solar heat, announced in a letter to *Nature*

that a clue had been found to explain the source of power of the Sun and the stars. The Curies had discovered that one gram of radium could supply 100 calories per hour, indefinitely and without itself cooling down to the ambient temperature. Wilson then computed that the amount of radium which would suffice to supply the Sun's entire output of energy. He found that a mere 3.6 grams per cubic metre would be sufficient.

Wilson's letter went unnoticed at first, but a few months later a note appeared in *Nature* that contained a similar speculation. This time, however, the author was George Darwin, the great biologist's son, who tried to support his father's idea of very long time periods. He was also a noted theoretical astronomer, and therefore his paper attracted immediate attention. Darwin discussed the implications of radioactivity for the Sun's heat. He reminded the readers that 50 years earlier Kelvin, in defending his upper limit of 400 million years for the age of the Earth, said in his 1862 paper: '*unless sources now unknown to us are prepared in the great storehouse of creation*' (Smith and Wise 1989, p 549). Darwin went on to explain: 'We have recently learned the existence of another source of energy such that the amount of energy available is so great as to render it impossible to say how long the Sun's heat has already existed, or how long it will last in the future' (Smith and Wise 1989, p 601).

Darwin concluded by reminding the reader that Kelvin's *concentrated Sun* model, essentially an extension of Helmholtz's constant density model, allowed an upper limit of 60 million years for the age of the Sun. Using Rutherford's measurements that a gram of radium could emit as much as 10^9 calories of heat, he estimated that, if the Sun were made of such radioactive material, it could emit 'nearly 40 times as much as the gravitational lost energy of the homogeneous Sun, and eight times as much as his concentrated Sun' (Smith and Wise 1989, p 599).

A simple calculation shows that actually about 2.2 g m^{-3} of the Sun's volume of radium could supply the energy output of the Sun. If 1 g of radium has a constant energy output of 100 calories per hour then the output is 0.116 J s^{-1} . Dividing the Sun's output (as known at that time) of $3.6 \times 10^{26} \text{ J s}^{-1}$ by 0.116 we obtain 2.2 g m^{-3} .

Finally, we will calculate the age of the Sun, based on Rutherford's estimate that 1 g of radium

has an energy content of about 1×10^9 calories, or $4.2 \times 10^{12} \text{ J}$. The mass requirement of the Sun is $3.6 \times 10^{26} / 4.2 \times 10^{12} = 8.57 \times 10^{13} \text{ kg s}^{-1}$. Dividing this quantity into the mass of the Sun we get about 740 million years, or, as Darwin claimed, '40 times as much as the gravitational lost energy of the homogeneous Sun'.

Darwin was pleased with the dramatic result that came from a simple 'back-of-an-envelope' calculation.

The embarrassing problem of having the age of the Sun smaller than the age of the Earth was thus solved. But only in the minds of the younger physicists and geologists. Kelvin was approached in 1906 by one of his former students, James Orr, who asked the following question: 'Do you agree, that if the Sun or Earth contains even a small amount of radioactive matter, all physical calculations of age and heat are overturned, and the old ratios of the geologists are restored?' (Smith and Wise 1989, p 610).

Kelvin seemed unmoved and did not concede. He remained committed to the primacy of the gravitational theory. There is, however, anecdotal evidence that privately Kelvin did concede. According to a conversation he had with J J Thomson in 1904, Kelvin apparently was willing to admit that 'radioactivity made some of the assumptions untenable' (Burchfield 1975, p 165). But no retraction was ever published.

Beyond radioactive energy

The discovery of radioactivity had two dramatic effects on the debate about the age of the Earth. First, it quickly became clear that since radioactive sources were found everywhere, including deep in the crust of the Earth, the heat budget of the Earth could not be reliably estimated. Secondly, by about 1910 the new methods of radioactive dating held the promise of finding a reliable way to determine the age of the Earth.

The source of the energy of the Sun, however, remained a mystery. By 1905 most physicists believed that radioactive energy must be somehow responsible for the awesome energy output of the Sun. Helium was known to be the byproduct of radioactivity and helium was found in the Sun. But there were objections: 'Why was Becquerel radiation (beta and gamma rays) absent from the solar spectrum?' And: 'Should they

not be emanating from the Sun?' 'Why did they not appear in the solar spectrum?' The physicist Robert Strutt, son of Kelvin's close friend Lord Rayleigh, showed that any beta or gamma radiation would be filtered out by the Sun. He also pointed out that the presence of helium in the Sun did give convincing evidence that the Sun was powered by radioactive energy.

The arguments for and against the Sun being powered by radioactive energy continued but by the 1920s the problem of the Sun's energy was overshadowed by the problem of finding reliable methods for dating the Earth and accounting for the heat energy observed coming from the Earth.

Block A.

A1. Newton's thought experiment

Newton must have made a quick mental calculation: if it takes about 1 hour for the 1 inch (2.54 cm) globe to cool to about room temperature from red hot, it would take a globe of the size of the Earth about 50 000 years to cool, because

$$T \sim \frac{12.8 \times 10^6}{2.54 \times 10^{-2}} \text{ hours}$$

which is a little more than 50 000 years. (The diameter of the Earth is approximately 12 800 km.)

A2. To find the cooling time for a hot sphere

Stefan's law of radiation states that the rate at which an object radiates heat per unit area is proportional to the fourth power of the absolute temperature:

$$H \propto T^4 \quad \text{or} \quad H = e\sigma T^4.$$

We now define Q as the total rate of heat energy leaving the body per unit time:

$$dQ/dt = e\sigma AT^4$$

where e is the emissivity of the object (between 0 and 1), σ is a universal constant equal to $5.7 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$, A is the surface area and T is the absolute temperature.

In order to find the approximate time for cooling we will assume that the ambient temperature T_f is 0 K. This will be almost true in space but not in a room. First we combine Stefan's law and the Joule energy content: $dQ = mc dT$

where c is the specific heat capacity and m the mass of the radiating object. Combining these equations and rearranging leads to

$$T^{-4} dT = (Ae\sigma/mc) dt.$$

Integrating both sides we obtain

$$\int_{T_i}^{T_f} T^{-4} dT = (Ae\sigma/mc) \int_0^t dt$$

where the temperature changes from T_i to T_f . We have

$$\frac{1}{3}(1/T_f^3 - 1/T_i^3) = (Ae\sigma/mc)t$$

or

$$t = (1/T_f^3 - 1/T_i^3)mc/3e\sigma A.$$

We are assuming that the conduction of heat through the metallic sphere is very rapid.

Applying this formula to the small sphere, we find that the cooling time is about 47 minutes. The sphere was assumed to have a diameter of $2.54 \times 10^{-2} \text{ m}$ and a density of $8.0 \times 10^3 \text{ kg m}^{-3}$.

To find the time of cooling of a globe of the size of the Earth that is made entirely of iron, we again assume that, as the globe cools, at any time the temperature of the large sphere is the same everywhere. The mass of the iron globe is $M = 8.7 \times 10^{24} \text{ kg}$, the surface area is $A = 5.14 \times 10^{14} \text{ m}^2$. Substituting into our formula we get about 45 000 years.

Block B.

B1. The life expectancy of the Sun made of coal

It was known that coal contained about $3.0 \times 10^7 \text{ J kg}^{-1}$ of potential energy of combustion. The energy output of the Sun was thought to be about $3.6 \times 10^{26} \text{ J s}^{-1}$ for the whole surface. The mass of the Sun was known to be about $2.0 \times 10^{30} \text{ kg}$. Therefore, the maximum life expectancy of the Sun would be

$$\frac{2.0 \times 10^{30} \times 3.0 \times 10^7}{3.6 \times 10^{26}} \text{ s} = \text{about 5000 years.}$$

B2. To determine the the gravitational potential energy of the Sun of constant density

Assume that the Sun grows by accretion of mass that falls in from far away, or infinity (figure 1). The density of the Sun is constant and at a distance

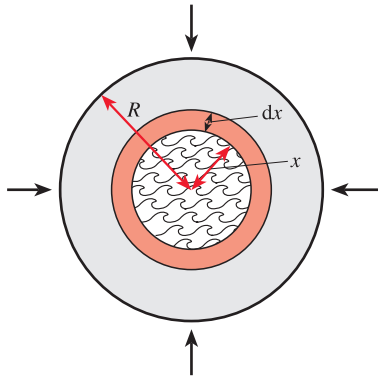


Figure 1. Helmholtz's gravitational contraction model: to find the age of the Sun.

x the Sun accumulates a mass dm . The amount of work done by the mass dm falling from infinity is

$$dW = \frac{GM_x dm}{x}.$$

Therefore, making use of $M_x = \frac{4}{3}\pi\rho x^3$ and $dm = 4\pi x^2\rho dx$ leads to

$$dW = \frac{16}{3}\pi^2 G\rho^2 x^4 dx.$$

Integrating (x changing from 0 to R) we get

$$W = \frac{16}{3}\pi^2 G\rho^2 \frac{R^5}{5}.$$

Using $\rho = 1.4 \times 10^3 \text{ kg m}^{-3}$ and $R = 7 \times 10^8 \text{ m}$ gives

$$W = 2.3 \times 10^{41} \text{ J}.$$

We can easily show that this is equivalent to the expression Helmholtz arrived at, by substituting the following:

$$\rho = M/V = M/\frac{4}{3}\pi R^3 \quad \text{and} \quad g = Gm/r^2$$

$$W = \frac{16}{3}\pi^2 G\rho^2 \frac{r^5}{5} = \frac{16}{3}\pi^2 G \left(\frac{M}{\frac{4}{3}\pi R^3} \right)^2 \frac{r^5}{5}$$

or, as Helmholtz wrote it,

$$V = \frac{3r^2 M^2 g}{5Rm}.$$

This result is best expressed in the compact form

$$V = \frac{3GM^2}{5R}.$$

B3. To estimate the age of the Sun

Since we know the energy output of the Sun S , we can estimate the age simply by dividing the gravitational potential energy by this energy output:

$$t = W/S = \frac{2.3 \times 10^{41}}{3.6 \times 10^{26}} \text{ s} \simeq 21 \text{ million years}.$$

B4. To find the temperature of the Sun

Following Helmholtz, we simply use the equation of Joule and relate it to the gravitational potential energy:

$$H = Mc\Delta T \quad \text{and} \quad W = AH.$$

Therefore

$$Amc\Delta T = \frac{3r^2 Mg}{5Rm}.$$

Solving for T (ΔT) we have

$$T = \frac{3r^2 M}{5ARmc} g.$$

This is the same result as Helmholtz's.

B5. To estimate the pressure at the centre of Sun

Only a very simple calculation is needed to estimate the pressure. One calculates the force produced by a 1 m^2 column of the Sun's gas, having a height of $7 \times 10^8 \text{ m}$, an average density of $1.4 \times 10^3 \text{ kg m}^{-3}$, and the average gravity being that of the middle, or $270/2 \text{ m s}^{-2}$, or 135 m s^{-2} .

$$\begin{aligned} F \text{ (on } 1 \text{ m}^2 \text{ area)} &= mg \\ &= 7 \times 10^8 \times 1.4 \times 10^3 \times 135 \\ &= 1.3 \times 10^{14} \text{ N m}^{-2}. \end{aligned}$$

B6. To determine the gravitational shrinkage of the Sun

A simple model would be the following. Assume that the Sun has a constant density of about $1.4 \times 10^3 \text{ kg m}^{-3}$. The gravity then will decrease linearly as we descend toward the centre of the Sun (see figure 2). Clearly, the whole mass of the Sun must be imagined to shrink. Assume that the top sinks by 1 metre. Then halfway down the shrinkage will be 0.5 m and in the centre zero. For our simple model then the work done for a 1 m

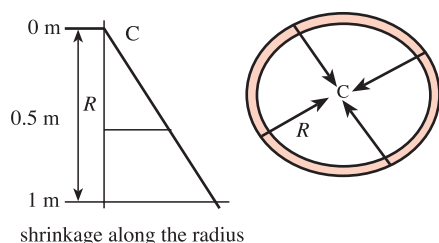


Figure 2. Helmholtz's gravitational contraction model: the energy of the Sun is produced by the shrinkage rate of the Sun.

shrinkage on top is equivalent to the work done by the whole mass of the Sun being concentrated at the midpoint where the gravity is half that on the surface and the shrinkage is 0.5 m. Since the gravity g_s on the surface is 270 m s^{-2} we have

$$W = M_s \times (g_s/2) \times 0.5 \\ = 2 \times 10^{30} \times (270/2) \times 0.5 = 1.35 \times 10^{32} \text{ J.}$$

Therefore, the time taken to fall through 1 m is $1.35 \times 10^{32} / 3.6 \times 10^{26} = 1.2 \times 10^{-2}$ years. This means that according to our simple model the Sun will contract about 84 m per year, or about 8.4×10^4 m per 1000 years. Could this magnitude of shrinkage be observed? This is only about 1/1000 of the radius of the Sun and could therefore not be observed. Helmholtz calculated the Sun's contraction rate to be 76 m per year, which is very close to our result.

Block C.

C1. Kelvin's meteoric theory

Kelvin calculated the effect of the additional mass of the Sun from meteoric impacts on the period of rotation of the Earth and found that it would be about 1/8 of a year in a 2000 year period. His calculations are quite sophisticated but we can check this value with a relatively simple calculation using Kepler's third law and Newtonian mechanics:

$$P^2 = (4\pi^2/GM)R^3.$$

We will simply compare the period of the Earth around the Sun 6000 years ago with the period today, keeping everything constant except the change in the mass of the Sun. We will take 365.0000 days as the year 6000 years ago. The

mass of the Sun changes from 1.0000 to 1.0002 units in that time. Therefore,

$$P_1/P_2 = (M_2/M_1)^{1/2}$$

$$P_2 = P_1 \times 1.0001 = 365.0365 \text{ days.}$$

Let us say that the average change is half of that, or 0.01825 days per year. The total change in that time then is 6000×0.01825 , or 110 days. For 2000 years then the change is 37 days, or 0.10 years. This is very close to Kelvin's value of 0.125 years.

C2. Kelvin's calculations to determine the age of the Earth

Kelvin used the well known equation of heat conduction:

$$\partial v/\partial t = k\partial^2 v/\partial x^2. \quad (\text{C.1})$$

Using Fourier integrals he showed that:

$$v = \frac{2V}{\pi^{1/2}} \int_0^{x/2\sqrt{kt}} dz e^{-z^2} \quad (\text{C.2})$$

where z is a dummy variable. By differentiating the above equation he obtained the temperature gradient:

$$\frac{dv}{dx} = \frac{V}{(\pi kt)^{1/2}} e^{-x^2/4kt} \quad (\text{C.3})$$

where k is the conductivity of the solid, V is the initial temperature (3700°C), x is the distance from the surface of the Earth, v is the temperature of the point x at time t , and dv/dx is the rate of variation of temperature per unit length, as we descend into the Earth.

Kelvin then plotted a graph that showed the thermal gradient and the temperature change as a function of the distance below the Earth's surface (see figure 3).

C3. A more detailed discussion of Kelvin's mathematics

The following problem can be found in standard advanced books on mathematical physics, and approximates Kelvin's model for the calculation of temperature in his article of 1862.

- A semi-infinite thin bar ($x \geq 0$) whose surface is insulated has an initial temperature equal to $f(x)$. A temperature of zero is suddenly applied (at $t = 0$) to the end $x = 0$ and is maintained. Find the temperature distribution as a function of t and x .

Using equation (1), applying boundary conditions and using a Fourier series, it can be shown that

$$v(x, t) = \frac{2V}{\pi^{1/2}} \int_0^{x/2\sqrt{kt}} dz e^{-z^2} = V \operatorname{erf}[x/2(kt)^{1/2}]. \quad (\text{C.4})$$

(Kelvin mentions in a footnote in his article that “A table of values of this integral, sometimes now called the ‘Error Function’, is to be found in Table III of de Morgan’s article on ‘The Theory of Probabilities’, *Encyclopedia Metropolitana*, Edition 1845, Vol. ii.”)

To solve for the temperature v , as we descend into the Earth’s interior we can use conventional tables of the ‘Error Function’, as Kelvin may have done. Or we can use the series solution of the erf:

$$\operatorname{erf}(x) = (2/\pi^{1/2})(x - x^3/3 + x^5/10 + x^7/42 \dots). \quad (\text{C.5})$$

But we are looking for the solution of

$$\operatorname{erf}(x/a)^{1/2} = (2/\pi^{1/2})[x/a - (x/a)^3/3 + (x/a)^5/10 \dots] \quad (\text{C.6})$$

where $a = 2(kt)^{1/2}$. Therefore

$$v(x, t) = V \operatorname{erf}[x/2(kt)^{1/2}] = (2/\pi^{1/2})V[x/a - (x/a)^3/3 + (x/a)^5/10 \dots]. \quad (\text{C.7})$$

The temperature v as a function of distance x can now be plotted and produces the graph shown in figure 3. Notice that the temperature v is almost linear up to $x = a$ (about 122 km). For example, the temperature at a depth of 12.2 km is simply:

$$12\,200/28^\circ\text{C}, \text{ or } 436^\circ\text{C}.$$

If we use equation (C.7) for the temperature, we obtain 439°C . However, at $x \simeq 1.5a$, the temperature begins to change very quickly as the ‘erf’ predicts.

A note of caution should be given at this point. In practice, tables of values for the error function erf should be used. The reason for this is that when x/a is greater than 1, it is difficult to evaluate the erf: the series does not converge easily and many terms must be evaluated to get a good result.

C4. The expression for the temperature gradient

Differentiate v with respect to x in equation (2), using the dummy variable $z = x/2(kt)^{1/2}$ to get equation (3). Using this equation, Kelvin then showed that for the temperature gradient to be about $1/28$ m per $^\circ\text{C}$, the time is about 10^8 years.

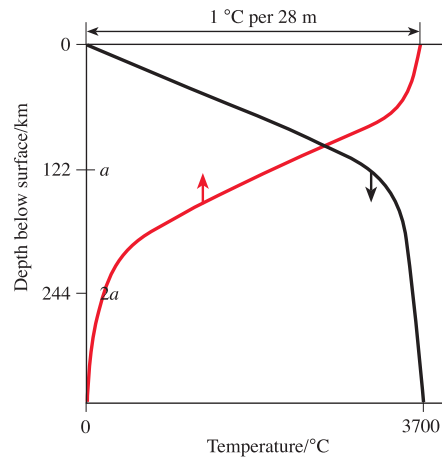


Figure 3. Kelvin’s graph of the cooling of the Earth for $t = 100\,000\,000$ years.

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References and further reading

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